Pre-Project Presentation

q-series, Ramanujan's theta functions and Overpartitions

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- John Wallis introduced hypergeometric series (book:Arithmetica Infinitorum): $1+a+a(a+b)+a(a+b)(a+2b)+\ldots+a(a+b)(a+2b)\ldots(a+(n-1)b)$ in 1655
- now for $b=1$ this series turns to $1+a+a(a+1)+a(a+1)(a+2)+\cdots+a(a+1)..(a+n-1)+..$ this was represented as $\sum_{n\geq 0}(a)_n$ where $(a)_n=a(a+1)\ldots(a+n-1),(a_0=1)$ by Leo August Pochhammer which was later called shifted factorial
- Leonhard Euler introduced the power series: ${}_2F_1\,(a,b,c,z)=1+\frac{ab}{1!c}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}z^2+..+\frac{a(a+1)...(a+n-1)b(b+1)...(b+n-1)}{n!c(c+1)...(c+n-1)}z^n+..$
- In 1812 Carl Friedrich Gauss reintroduced this in his famous paper as a solution to 2nd order differential equation:

$$x(x-1)y''+[c-(a+b+1)x]y'-aby=0 \tag{1}$$

Origins and History

- From this infinite series many other series can generalized like: other series solutions of ODEs, harmonic series etc

  (by Riemann's theorem : if $a(x)y'' + b(x)y' + c(x)y = 0$ has only 3 regular singularities then it is equivalent to equation (1))
- a generalized hypergeometric function has a series representation $\sum_{n=0}^{\infty} c_n$ with c_{n+1}/c_n a rational function of n eg : $c_n = (a)_n x^n/(b)_n$
- Basic hypergeometric series are series $\sum_{n\geq 0} c_n$ with c_{n+1}/c_n a rational function of q^n for a fixed parameter q, Euler proved many identities using these type of series
- influenced by Euler's work on continued functions Heinrich Eduard Heine introduced

  basic Hypergeometric series : ${}_2\phi_1(q^a, q^b; q^c, q, z) = \sum_{n=0}^{\infty} \frac{(q^a;q)_n\,(q^b;q)_n}{(q;q)_n\,(q^c;q)_n} z^n$

  where $(a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1})$

  on observing that $\lim_{q\to 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n$ so as $q \to 1$ $\quad {}_2\phi_1 \to {}_2F_1$

Hypergeometic series

- we define ${}_rF_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s) = 1 + \sum_{n=1}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n}{n!(b_1)_n (b_2)_n \dots (b_s)_n} z^n$ where generally z is a complex number and b_i's are defined such that the denominator is not zero
- This series terminates if one of $a_i = -n$ i.e. a negative integer
- This series converges:
  - if $r \leq s$ for all z
  - if $r = s + 1$ then for all $|z| < 1$ and
  - if $|z| = 1$ and $r = s + 1$ then when $Re[b_1 + b_2 + \dots + b_s - (a_1 + a_2 + \dots + a_r)] > 0$
  - diverges otherwise
- Eg: ${}_2F_1(a, b; c, z)$ converges: for all $|z| < 1$ and if $Re(c - a - b) > 0$ for $|z| = 1$ diverges for all $|z| > 1$

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for all $|z| \leq 1$ and $c \neq -n$, $n \in \mathbb{N}$ we have

Binomial Theorem

$$ {}_2F_1(a, c; c; z) = {}_1F_0(a; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = \frac{1}{(1-z)^a} $$

Euler's Transformation

$$ {}_2F_1(a, b; c; z) = (1-z)^{c-a-b} \, {}_2F_1(c-a, c-b; c, z) $$

Gauss's Summation

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

- Going with the same spirit as hypergeometric series we define
$${}_2\phi_1(a,b;c,q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n}{(q;q)_n\,(c;q)_n} z^n$$
In general
$${}_r\phi_s(a_1,a_2,\dots,a_r;b_1,b_2\dots,b_r;q,z) = \sum_{n=0}^{\infty} \frac{(a_1,a_2,\dots,a_r;q)_n}{(q;q)_n(b_1,b_2,\dots,b_s;q)_n} \left[(-1)^n\, q^{\left(\frac{n(n-1)}{2}\right)}\right] z^n$$
where b_i's are such that denominator is not zero and $\left[(-1)^n\, q^{\left(\frac{n(n-1)}{2}\right)}\right]$ is used to simplify calculations
- This series terminates if one of $a_i = q^{-n}$ for some positive integer n
- This series converges if : $r \leq s$ and $|q| < 1$ for all z or if $r = s+1$ and$|z| < 1$ or if $|q| > 1$ and $|z| < |\frac{b_1 b_2 \dots b_s q}{a_1 a_2 \dots a_r}|$ diverges otherwise

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q-analogues of Hypergeometic series Identities

Binomial Theorem

$$ {}_1\phi_0(a;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty} $$

Euler's Transformation

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(abz/c;q)_\infty}{(z;q)_\infty}\, {}_2\phi_1(c/a, c/b; c, q, abz/c) $$

Gauss's Summation

$$ {}_2\phi_1(a,b,c;q,c/ab) = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty} $$

q-series Identities

Heine's Transformation

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty}\, {}_2\phi_1(c/b,z;az;q,b) $$

Bailey's Transformation

$$ {}_2\phi_1(a,b;ab/q,q,-q/b) = \frac{(aq,aq/b^2;q^2)_\infty(-q;q)_\infty}{(aq/b,-q/b;q)_\infty} $$

Jacobi's triple product

$$ (q^2;q^2)_\infty(-zq;q^2)_\infty(-q/z;q^2)_\infty = \sum_{n=-\infty}^{\infty} z^n q^{n^2} $$

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Partitions

- Partition : for a non negative integer n a partition of n is a non increasing sequence of positive integers whose sum is n for eg: (3 2 1) is a partition of 6 as 3+2+1=6
- Number of possible partition of n is denoted by $p(n)$ (p(0)=0 by convention)
  eg: p(5)= 7 as (5),(4 1),(3 2),(3 1 1),(2 2 1),(2 1 1 1),(1 1 1 1 1) are partitions of 5
- many mathematicians studied partitions but not until Euler who found its generating functions , accelerating the development of the field
- Partition theory finds its application in probability, statistics, combinatorics and particle physics
- Prominent Mathematicians like Cayley, Gauss, Hardy, Jacobi, Lagrange, Legendre, Littlewood, Rademacher, Ramanujan, Schur, and Sylvester have contributed to the development of the theory

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q-series and Partition functions

for General partition function p(n):

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}$$

for distinct partition function $p_d(n)$:

$$\sum_{n=0}^{\infty} p_d(n)q^n = (-q;q)_\infty$$

if $p(N,M,n)$ denote the number of partitions on n into at most M parts each $\leq N$ then:

$$G(N,M;q) = \sum_{n\geq 0} p(N,M,n)q^n = \frac{(q;q)_{N+M}}{(q;q)_N (q;q)_M}$$

Ramanujan's theta functions

- $f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2}$
- special cases :

$$\begin{aligned}
\varphi(q) = f(q,q) &= \sum_{n=-\infty}^{\infty} q^{n^2} \\
\psi(q) = f(q,q^3) &= \sum_{n=0}^{\infty} q^{n(n+1)/2} \\
f(-q) = f(-q,-q^2) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} \\
\chi(q) = (-q;q^2)_\infty &
\end{aligned}$$

Theta function identities

- $f(a,b) = f(b,a), \quad f(1,a) = 2\varphi(a), \quad f(-1,a) = 0$
- if $r_k(n) and t_k(n)$ are number of ways to represent n as sum of k squares and as sum of k triangular numbers respectively then:

$$\varphi^k(q) = \textstyle\sum_{n=0}^{\infty} r_k(n)q^n$$
$$\psi^k(q) = \textstyle\sum_{n=0}^{\infty} t_k(n)q^n$$

- Jacobi's triple product:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

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$$\varphi(q) = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \qquad \psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \qquad f(-q) = (q;q)_\infty,$$
$$\chi(-q^2) = \chi(q)\chi(-q)$$

Theorem

for any positive integer n

$$p(5n+4) \equiv 0 \pmod 5$$

Proof.

- for any sequence of integers a_n, $L(q) = \frac{\sum_{n\geq 0} a_n q^{n^2}}{(q;q)_\infty^2}$ the coefficient of q^{5n+3} in $L(q)$ is divisible by 5:
  - take $L(q) = (q;q)_\infty^3 \frac{\sum_{n\geq 0} a_n q^{n^2}}{(q;q)_\infty^5} \equiv (q;q)_\infty^3 \frac{\sum_{n\geq 0} a_n q^{n^2}}{(q^5;q^5)_\infty} \pmod 5$
  - now as $(q^5, q^5)_\infty$ term gives rise only to 5 divisible terms it is enough to examine for q^{5n+3} terms in

    $(q;q)_\infty^3 \sum_{n\geq 0} a_n q^{n^2} = \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2} \sum_{n\geq 0} a_n q^{n^2}$ by Jacobi's identity
  - so for our condition the coefficient terms of q power $j(j+1)/2 + m^2 = 5n+3$, $j(j+1)/2 + m^2 = 5n+3 \iff ((2j+1)^2 - 1)/8 + m^2 - 3 = 5n \equiv 0 \pmod 5$

$\square$

Proof.

$\iff ((2j+1)^2 - 1 + 8m^2 - 24)/8 \equiv 0 \pmod 5 \iff (2j+1)^2 + 8m^2 - 25 \equiv 0 \pmod 5 \iff (2j+1)^2 + 3m^2 \equiv 0 \pmod 5$

- since $(2j+1)^2 \equiv 0, 1, 4 \pmod 5$ and $3m^2 \equiv 0, 2, 3 \pmod 5$ only $\implies (2j+1)^2 + 3m^2 \equiv 0 \pmod 5 \iff 2j+1 \equiv 0 \pmod 5$
- now coefficients q^{5n+3} are $(-1)^j(2j+1)a_n$ is divisible by 5
- now $\frac{1}{(q^2;q^2)_\infty} = \sum_{n\geq 0} p(n)q^{2n} = \frac{1}{(q;q)_\infty(-q;q)_\infty} = \frac{(q;q)_\infty}{(q;q)_\infty^2(-q;q)_\infty} = \frac{1}{(q;q)_\infty^2}\varphi(-q) = \frac{1}{(q;q)_\infty^2}(1 + 2\sum_{m=1}^{\infty}(-1)^m q^{m^2})$
- thus for $2k = 5n+3$ terms the coefficients are $\equiv 0 \pmod 5$, now $2k = 5n+3 \iff k \equiv 5n+4$

$\square$

Overpartitions

- Many proofs of theorems like q-binomial theorem, Heine's transformation, Lebesgue's identity, Ramanujan's ${}_1\psi_1$ summation and q-Gauss summations involve a naturally recurring quantity that is more general than partitions: Overpatitions which was discussed MacMohan's combinatorics analysis.
- So the study of these overpartitions make sense, theory of basic hypergeometric series contains a wealth of information about overpartitions and many theorems and techniques of ordinary partitions have analogues for overpartitions
- Making use of already developed partition theory a generating q-series function for overpartition is developed with some restrictions

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Overpartition functions

- An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence (equivalently, the final occurrence) of a number may be overlined
- for example overpartitions of 3 are $3, \bar{3}, 2+1, \bar{2}+1, 2+\bar{1}, \bar{2}+\bar{1}, 1+1+1, \bar{1}+1+1$
- if $\bar{p}(n)$ indicate the number of overpartition of n then:

$$\sum_{n=0}^{\infty} \bar{p}(n) q^n = \frac{(-q,q)}{(q,q)} \\ = \frac{1}{\varphi(-q)}$$

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Theorem

if $\bar{p}_o(n)$ represents the number of overparrtiions of n in which only odd parts occur then:

$$\bar{p}_o(n) = \begin{cases} 2 \ mod\ 4 & \textit{if } n \textit{ is square of twice a square number} \\ 0 \ mod\ 4 & \textit{otherwise} \end{cases}$$

Proof.

- if $\bar{P}(n) = \sum_{n\geq 0} \bar{p}(n)q^n$ and $\bar{P}_o(n) = \sum_{n\geq 0} \bar{p}_o(n)q^n$ then

$$\bar{P}(n) = \varphi(q)\bar{P}(q^2)^2$$
$$\bar{P}_o(n) = \varphi(q)\bar{P}(q^2)$$

- using this recurrence formula we get $\quad \bar{P}_o(n) = \varphi(q)\varphi(q^2)\varphi(q^4)^2\varphi(q^8)^4\ldots$
- now $\quad \varphi(q)^{2^k} \pmod 4 = (1 + 2\sum_{n\geq 1} q^{n^2})^{2^k} \pmod 4 = 1 \pmod 4$ for $k \geq 1$ so

$$\implies \bar{P}_o(n) \pmod 4 = \phi(q)\phi(q^2) \pmod 4 = (1 + 2\sum_{n\geq 1} q^{n^2})(1 + 2\sum_{n\geq 1} q^{2n^2})(mod\ 4)$$
$$= 1 + 2\sum_{n\geq 1} q^{n^2} + 2\sum_{n\geq 1} q^{2n^2} \pmod 4$$

$\square$

Future plans

- Studying more of congruence relations in overpartitions
- Studying l-regular and singular overpartitions and theirs generating functions and relations
- Deriving more of congruence relations after the study
- Studying colored partitions and their relations

References

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