# Numerical Linear Algebra

## Yashas.N

if a matrix is in triangular form one can easy calculate its inverse by making note that inverse of a triangular matrix is of the same triangular type i.e. for example A is upper triangular non-singular matrix then  $A^{-1}$  is also an upper triangular matrix.

## LU Decomposition

From previous point we have if any non singular matrix A can be written as A = LU for lower triangular L and upper triangular U then  $A^{-1} = U^{-1}L^{-1}$  thus inverse can be easily calculated.

■ This Decomposition may not be unique ■ To decompose in a easy way we take diagonal elements of U or L as 1. (only in one of the factors) and compute the coefficients by writing A = LU and solving some equations in a linear order.

■ Now in addition if principal minors  $(\Delta_k)$  of matrix A are not zero then the above decomposition is unique.

### Gauss elimination

if  $A = [a_{ij}]$  be a  $n \times n$  non singular matrix then for linear system Ax = b then we can use elementary operations:

exchange of rows, addition of rows and multiplication by a non zero constant to a row to transform the linear systemA'x = b' such that  $a'_{11} \neq 0$  and  $a'_{11} = 0$  for i < 1 and continuing this process to get for i = 2, 3, ..., nwe get a system  $Gx = \tilde{b}$  where G is upper triangular and has same solutions as original system.

## Gauss-Jordan method

this method is similar to Gauss elimination but Ax = b for non singular square A is transformed to  $G_Jx = \tilde{b}$  where  $G_J$  is diagonal i.e. for  $A = [a_{ij}]$ ,  $a_{ii}$  is made non zero and all other  $a_{ij}$  is made zero with elementary transformations.

#### General Iterative methods

iterative methods can be generalised as  $x^{(k)} = Tx^{(k-1)} + c$ 

■ this method converges to a unique solution for any initial approximation  $x^{(o)}$  iff  $(\iff) \rho(T) < 1$  where  $\rho(T) = \max(|\lambda|)$  for  $\lambda$  eigenvalue of T.

#### Jacobi's Method

■ if 
$$Ax = b$$
 is a system such that for n-  
square  $A = [a_{ij}]$  we have  $a_{ii} \neq o$  (if not is  
made by rearranging rows or equations if possible)  
then for  $x = [x_i]^T$  we can transform  $x_i = \begin{bmatrix} \sum_{\substack{j=1 \\ j\neq i}}^{n} (-a_{ij}x_j/a_{ii}) + b_i/a_{ii} \end{bmatrix}$  from which we

get the iterative method i.e.  $x^{(o)}$  is initial approximation and for  $k^{th}$  approximation  $x^{(k)}$  we have the iteration using  $x^{(k-1)}$  given by

$$x_i^{(k)} = \frac{\mathbf{1}}{a_{ii}} \left[ \sum_{\substack{j=1\\ j \neq i}}^n (-a_{ij} x_j^{(k-1)}) + b_i \right].$$

■ Now for matrix representation if A = D + L + U where D is diagonal L is lower diagonal with diagonal entries o and U is

upper diagonal with diagonal entries o then for Jacobi method we have

$$\begin{split} (D+L+U)x &= b \\ \implies Dx = -(L+U)x + b. \\ \implies x = -D^{-1}(L+U)x + D^{-1}b. \\ \text{i.e.} \quad x^{(k)} &= -D^{-1}(L+U)x^{(k-1)} + D^{-1}b \end{split}$$

so we get  $T = -D^{-1}(L + U)$ ,  $c = D^{-1}b$  for general form.

## Gauss-Seidel Method

■ This is similar to Gauss method but here we use the previous  $k^{th}$  iterated variables for the next  $k^{th}$  one i.e. for in  $x_i^{(k)}$  iteration we can replace  $x_j^{(k-1)}$  for j < i with  $x_j^{(k)}$  as these are already found i.e.  $x_i^{(k)} =$ 

$$\frac{1}{a_{ii}} \left[ -\sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^n a_{ij} x_j^{(k-1)} + b_i \right].$$

■ for matrix representation we rewrite the iterative formula as

$$\sum_{j=1}^{n} a_{ij} x_j^{(k)} = -\sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} + b_i$$
  
similar to Jacobi's case if  $A = D + L + U$  by  
above formula we have

$$\begin{split} (D+L)x^{(k)} &= -Ux^{(k-1)} + b.\\ \text{i.e.} x^{(k)} &= -(D+L)^{-1}Ux^{(k-1)} + (D+L)^{-1}b.\\ \text{so we get } T &= -(D+L)^{-1}U, c = (D+L)^{-1}b. \end{split}$$

for system Ax = b, A = D + L + U
if A is strictly diagonal then both Jacobi and Gauss-Seidel methods converge for every initial approximation x<sup>(o)</sup>.
Gauss-Seidel method is twice as fast as Jacobi's method for convergence now from general iterative methods we have
sufficient condition for convergence of Ja-

cobi's method is that

$$\|T\| = \|-D^{-1}(L+U)\| < 1 \quad \text{i.e. } \rho(T) < 1.$$

■ similarly sufficient condition for convergence of Gauss-Seidel method is that

$$\|T\| = \| - (D + L)^{-1} U\| < 1.$$

■ Both these method also converge if  $A = [a_{ij}]$  is such that

$$\sum_{\substack{j=1\\ j\neq i}}^{n} |\mathfrak{a}_{ij}| \leqslant |\mathfrak{a}_{ii}| \text{ for } i = 1, 2, \dots, n \text{ and strict in-}$$

equality holds for at least one i.

# 0 References

- Burden R. L., Faires D. J., Burden A. M.: Numerical Analysis, Cengage Learning,(2016).
- [2] S. S. Sastry: Introductory Methods of Numerical Analysis, PHI Learning,(2012).